

Domination and Independence Numbers For Cylindrical Honeycomb Chessboards

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Abstract

What's the fewest number of queens on a chess board that attack every square? What's the largest number of non-attacking queens that can be placed on the board? Our research focuses on answering these questions for the five main chess pieces on a cylindrical honeycomb chess board. The answers to these questions are known as the domination and independence numbers, respectively, of the associated graphs.

Introduction

The age and popularity of chess has made it an appealing topic of study for mathematicians. In particular, the diversity of piece-movement has led to the development of several popular questions, including the following. What is the fewest number of queens that can be placed on the board so that every square is attacked by at least one queen? What is the maximum number of queens that can be placed on the board in such a way that no queen is attacking another? These same questions can also be asked about each of the other five chess pieces. Such questions are interesting because they represent the fundamental offensive and defensive capabilities for the pieces.

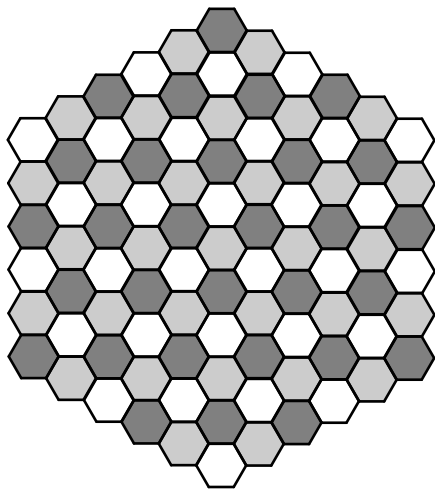


Figure 1. Gliński's hexagonal-chess board B_{11}

Gliński hexagonal chess, first created by Władysław Gliński, is a chess variant played on a hexagon-tiled board. The vertical board contains 91 hexagonal cells across 11 files, which is defined this as B_{11} . The cells are colored in three shades - light, dark, and midtone - with the central cell being midtone.

Gameplay follows the standard rules of chess, however there is one extra bishop and pawn per player. Pieces move and capture identically to the standard 8x8 board. The hexagonal board opens new possibilities, while players we still be able to keep the strategy of the classic game. Since its invention, hexagonal chess has emerged as one of the most popular alternate forms of chess. The new board geometry presents players with fresh challenges and has inspired new lines of analysis and theory.

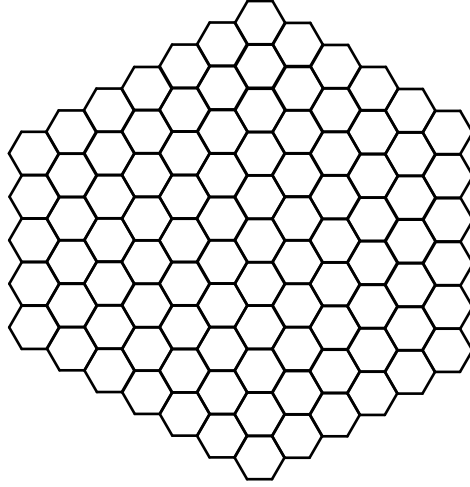


Figure 2. Cylindrical Hexagonal Board CB_{11}

Notation and terminology

Throughout this paper we will refer to the themes illustrated in this section. We will denote names of the boards as Q_n , K_n , R_n , B_n , and N_n for the queen, king, rook, bishop, and knight. On a cylindrical hexagonal board we will denote the names as CQ_n , CK_n , CR_n , CB_n , and CN_n . Respectively on a Hexagonal board of side n . Vertices are represent as hexes of the board. If a piece on a vertex can move to another hex, that hex is consider adjacent.

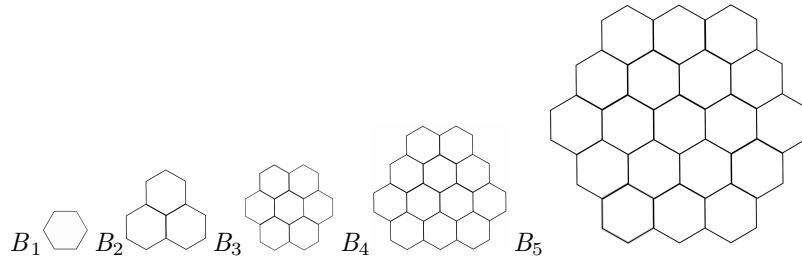


Figure 3. Hexagonal hexagon boards.

Taken from [3] to determine the size of B_n , B_1 is one hexagon, and if B_2 has three hexagons with a common vertex, then B_n for $n \geq 3$ consists of B_{n-2} together with all neighboring hexagons of B_{n-2} .

A if each vertex(hexagons), V , is adjacent to a vertex. It dominates itself and its neighbors. The domination number of a graph G , $\gamma(G)$, is the minimum

cardinality of a dominating set. In game terms how many pieces can I put on the board where I threaten or occupy every hexagon(vertex) on the board with the smallest number of pieces.

The independence number of a graph G , $\beta_0(G)$, is the maximum cardinality of an independent set of G . In game terms it is to place the maximum number of pieces so that no piece are threatened. Domination and independence are exactly the concepts needed to attack our triangular chessboard problems.

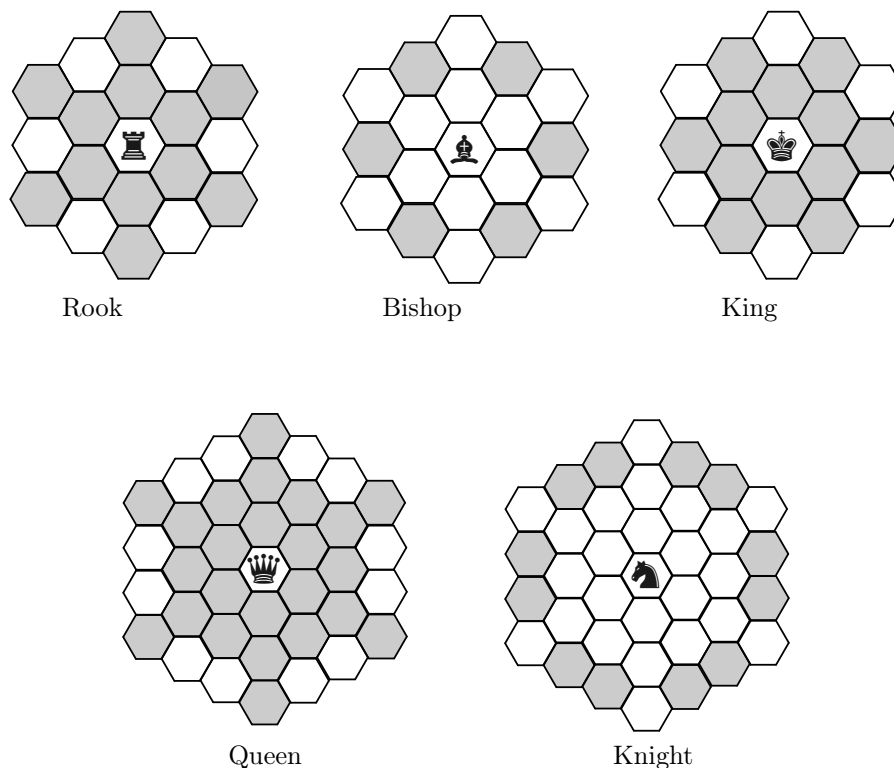


Figure 2. Moves on a Hexagonal Chessboard

Analysis of the Of Pieces

On a hexagonal chessboard, the independence numbers of all the pieces have already been found in [1]. Rook domination has also been found in [2]. When incorporating the cylindrical wrapping of boards, for an odd board, when $n \not\equiv 0 \pmod{2}$, an extra even column is needed on either side. For even board, when $n \equiv 0 \pmod{2}$, no extra column is needed because one border has an odd number and the other has an even, see Figure 3. To represent this mathematically

taken from [2], at the border of B_n where x represents the border, $x = \frac{n+1}{2}$ for odd boards. For even boards, $x = \frac{n+2}{2}, \frac{n}{2}$. To represent the extra column for odd boards where $k =$ new column, $k = \frac{n+1}{2} - 1$.

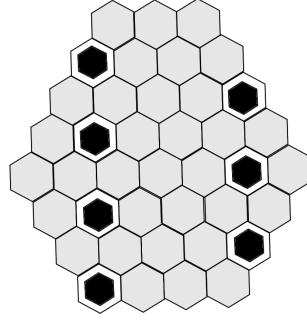


Figure 3: $\beta(CR_7)$

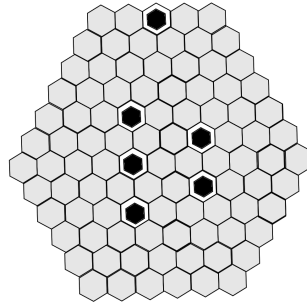


Figure 4: $\gamma(CR_{11})$

The Rook, CR_n , On boards that were even, the $\beta(CR_n) = \beta(R_n)$ and $\gamma(CR_n) = \gamma(R_n)$. For boards that were odd, the $\beta(CR_n) = \beta(R_n)$ and the $\gamma(CR_n) = \gamma(R_n) + 1$. Due to the new column being odd, for $\gamma(CR_n)$ it leaves one center piece that isn't attacked by the other pieces, see Figure 4.

The King, CK_n we have found that the $\beta(CK_n) = \beta(K_n) + \lceil \frac{j}{2} \rceil$ where $j = k = \frac{n+1}{2} - 1$ (Note $k =$ new border)

We have also found bounds for the $\beta(b_n)$ respective to figure 1, we were able to put 3 bishops on the board meaning. That the $\beta(b_n) = 9 \leq x \leq 11$

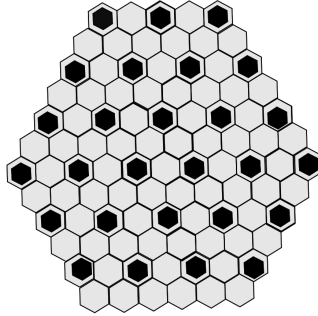


Figure 5: $\beta(CK_{11})$

Reflection

Currently the project is uncompleted, however With more allocated time we will be able to come up with conjectures and proofs by construction. As well as More bounds for certain boards

References

- [1] Bode, J.-P., Harborth, H. (2000). Independent chess pieces on Euclidean boards. *JCMCC. The Journal of Combinatorial Mathematics and Combinatorial Computing*, 33.
- [2] Wagon, S. (2014). Graph Theory Problems from Hexagonal and Traditional Chess. *The College Mathematics Journal*, 45(4), 278–287. doi:10.4169/college.math.j.45.4.278
- [3] DeMaio, J., Tran, H. L. (2013). Domination and Independence on a Triangular Honeycomb Chessboard. *The College Mathematics Journal*, 44(4), 307–314. doi:10.4169/college.math.j.44.4.307
- [4] Harborth, H., Nienborg, H. (2022). Rook domination on hexagonal hexagon boards. *Number Theory and Combinatorics*, 259–266. doi:10.1515/9783110754216-014
- [5] Bode, J.-P., Harborth, H., Kultan, V. (2006). Independent bishops on hexagon and triangle boards. *Congressus Numerantium*.